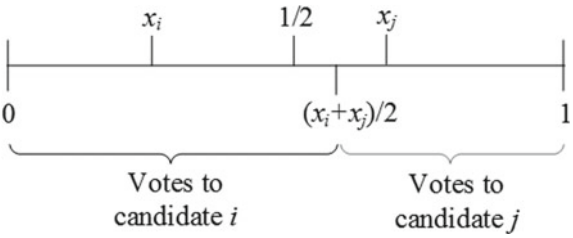

Fig. 4.12 Votes to candidate i and j when $x_i < x_j$ and $x_i > 1 - x_j$

we know that if, in addition, condition $x_i > 1 - x_j$ holds, candidate i wins the election,[9] as depicted in Fig. 4.12. In this context, candidate j would have incentives to deviate (e.g., moving x_j downward closer to x_i). If, instead, condition $x_i > 1 - x_j$ is violated, candidate i loses the election, implying that it is now this candidate who has incentives to deviate upward (moving closer to x_j). Therefore, asymmetric policy profiles cannot be sustained in equilibrium.

Second, we can consider symmetric policy announcements where both candidates announce the same policy, $x_i = x_j = x$. If $x < \frac{1}{2}$, the mass of voters to the right of x is larger than that to its right, entailing that one candidate receives more votes and wins the election, providing incentives to her rival to deviate from $x < \frac{1}{2}$. A similar argument applies, of course, if $x > \frac{1}{2}$.

However, at $x = \frac{1}{2}$, both candidates receive the same number of votes, and the winner is randomly selected. In this context, the loser of the random assignment does not have incentives to unilaterally deviate, as this deviation (above or below x) would decrease her votes below 50%, losing the election with certainty. Therefore, $x_i^* = x_j^* = \frac{1}{2}$ is a NE of the electoral competition game, because no candidate has incentives to deviate from her policy announcement. When one allows for three or more candidates running for office, however, no NE exists, as shown in Cox (1987) and Chisik and Lemke (2006).

Exercises

4.1. **Cournot duopoly with asymmetric marginal costs.**^{A} Consider the Cournot duopoly in Sect. 4.2.1, but assume that firm 1 faces marginal cost c_1 while firm 2's is c_2, where $c_1 < c_2 < 1$ (so firm 1 enjoys a cost advantage relative to firm 2).

(a) Find the best response function of firm 1 and of firm 2. Compare them.

[9] Recall that candidate i receives more votes than j does, thus winning the election, when $\frac{x_i+x_j}{2} > 1 - \frac{x_i+x_j}{2}$, which simplifies to condition $x_i > 1 - x_j$ or, alternatively, to $\frac{x_i+x_j}{2} > \frac{1}{2}$. In the context of Fig. 4.12, the last inequality indicates that the midpoint between x_i and x_j, $\frac{x_i+x_j}{2}$, lies to the right of the center of the line, 1/2. If, in contrast, $x_i < 1 - x_j$, the midpoint would lie to the left of 1/2, and candidate j would win the election.

(b) Insert firm 2's best response function into firm 1's to find the output that each firm produces in the Nash equilibrium of the Cournot game of quantity competition. Which firm produces a larger output?
(c) *Symmetric firms.* Assume that both firms now become cost symmetric, so that, $c_1 = c_2 = c$. Evaluate your results from part (b) at $c_1 = c_2 = c$, showing that you obtain the same results as in Sect. 4.2.1.

4.2. **Coordinated production decisions with differentiated products.**^{B} Consider a duopoly where every firm i faces inverse demand function

$$p(q_i, q_j) = 1 - q_i - \gamma q_j$$

where $\gamma \in [0, 1]$ denotes how sensitive firm i's price is to its rival's sales. When $\gamma = 0$, both firms are unaffected by each other's sales, as if they were two separate monopolists; whereas when $\gamma = 1$, firms sell homogeneous goods. For simplicity, assume that both firms face the same marginal production cost, c, where $1 > c \geq 0$.
(a) *Maximizing individual profits.* Find firm i's best response function, $q_i(q_j)$.
(b) How is the best response function affected by an increase in parameter γ? Interpret.
(c) Find the equilibrium output q_i^* and profits, π_i^*.
(d) How are equilibrium output and profits affected by an increase in parameter γ? Evaluate your equilibrium output and profits in the extreme cases where $\gamma = 0$ and $\gamma = 1$. Interpret.
(e) *Maximizing joint profits.* Assume that both firms could coordinate their output decision to maximize their profits. Find their equilibrium output and profits in this context.
(f) How are output and profits that you found in part (e) affected by an increase in parameter γ? Evaluate your equilibrium output and profits in the extreme cases where $\gamma = 0$ and $\gamma = 1$. Interpret.

4.3. **Cournot competition with cost externalities.**^{B} Consider a Cournot duopoly game where every firm i's inverse demand is $p(Q) = 1 - Q$, and Q denotes aggregate output. Every firm's marginal cost is, instead of c, given by $c - \alpha q_j$, where $\alpha \in [-1, 1]$ measures the degree of cost externalities that firm j's output imposes in firm i. This setting allows for cost externalities to be positive, $\alpha > 0$, if firm j's production decreases its rival's marginal cost; and negative, $\alpha < 0$, if q_j increases firm i's marginal cost.
(a) Find every firm's best response function. How is it affected by parameter α? Interpret.
(b) Find every firm's equilibrium output, aggregate output, price and profit. How are they affected by parameter α? Interpret.

4.4. **Mergers with cost externalities.**^{B} Consider the setting in Exercise 4.3 about firms competing in quantities and facing cost externalities. Assume that firms merge into a single entity that maximizes joint profits.
(a) Find individual output and aggregate output in equilibrium.

(b) Does the merger reduce aggregate output under all values of parameter α? If not, when does it increase aggregate output? Interpret in terms of merged firms internalizing their cost externalities.
(c) Find equilibrium price and profits.
(d) Does the merger increase price and profits?
(e) For which values of α does the merger increase both profits and aggregate output? For which values it only increases one of them? For which values it decreases both?

4.5. **Cournot competition with only one firm benefiting from a cost advantage.**^{B} Consider an industry like that in Sect. 4.2.2, namely, $N \geq 2$ firms selling a homogeneous good, competing á la Cournot, with inverse demand function $p(Q) = 1 - Q$, where Q denotes aggregate output. One of the firms (firm 1) faces marginal cost $c \in (0, 1)$, while its $N - 1$ rivals face a higher marginal cost c', where $1 > c' \geq c$. This cost advantage may be explained because firm 1 had more years of experience in the industry or operates in similar markets.
(a) Find firm 1's best response function and the best response function of each of its rivals.
(b) Find equilibrium output levels for each firm.
(c) Evaluate the equilibrium output for firm 1 and its $N - 1$ rivals when all firms are symmetric, that is, $c = c'$.
(d) Under which conditions on firm 1's cost, c, will all firms produce a positive output in equilibrium? Under which conditions will only firm 1 be active? Interpret.
(e) *Symmetric firms.* Evaluate your results in part (d) in the case that all firms are symmetric, that is, $c = c'$.

4.6. **Bertrand competition with asymmetric costs.**^{A} Consider the Bertrand model with two firms competing in prices, see Sect. 4.3.1. Assume now that firm 1's marginal cost is c_1, while that of firm 2 is c_2, where $c_1 < c_2 < 1$.
(a) Find a pure-strategy NE in this game.
(b) Evaluate equilibrium profits in your results from part (a).
(c) Consider now that the industry has $N \geq 2$ firms, where firm 1's marginal cost, c_1, is the lowest, that is, $c_1 < c_2 \leq c_3 \leq ... \leq c_N$. Find a pure-strategy NE in this setting.

4.7. **Three firms competing in prices and selling differentiated products.**^{C} Consider three firms competing à la Bertrand selling heterogeneous goods with an inverse demand function

$$p_i(q_i, q_j, q_k) = 1 - q_i - \beta(q_j + q_k),$$

where parameter β denotes the degree of product differentiation. When $\beta = 0$ products are completely differentiated, but when $\beta = 1$ products are homogeneous. For simplicity, assume that marginal costs are normalized to zero.
(a) Find every firm i's demand function. [*Hint*: Solve for q_i.]

(b) Find every firm i's best response function.
(c) Find the equilibrium price of every firm i. How does it change with β? Interpret.
(d) Find the equilibrium output of every firm i. How does it change with β? Interpret.

4.8. **Public good game with $N \geq 2$ players.**^{B} Extend the public good game in Sect. 4.4 to $N \geq 2$ players. For simplicity, assume that all players have the same wealth, w.
(a) Find player i's best response function $x_i(X_{-i})$, where $X_{-i} = \sum_{j \neq i} x_j$ denotes the aggregate donations by all player i's opponents.
(b) Find equilibrium donation x_i^*.
(c) How are equilibrium donations affected by a marginal increase in N? Interpret.
(d) Evaluate your equilibrium results at $N = 2$. Show that your equilibrium results coincide with those in Sect. 4.4.
(e) How are aggregate donations, X^*, affected by a marginal increase in N? Interpret.

4.9. **Socially optimal contributions with $N \geq 2$ players.**^{B} Consider the setting in Exercise 4.8.
(a) If players could coordinate their donations to maximize their joint utility, what would their donations be?
(b) Compare your socially optimal results with those in equilibrium.

4.10. **Public good game with quasilinear utility.**^{A} Consider the public good game in Sect. 4.4, but assuming that every player i's utility function is quasilinear, as follows

$$u_i(x_i, x_j) = (w_i - x_i) + \sqrt{mX}.$$

(a) Find player i's best response function $x_i(x_j)$. How does it depend on player i's wealth, w_i?
(b) Find the equilibrium donation x_i^*.
(c) How are equilibrium donations affected by a marginal increase in w_i or w_j? Are they affected by parameter m? Interpret.
(d) Find aggregate donations, X^*. Are they affected by a marginal increase in w_i or w_j? Interpret.
(e) *Socially optimal outcome.* If players could coordinate their donations to maximize their joint utility, what would their donations be?

4.11. **Testing for inefficiency in public good games with asymmetric wealth.**^{B} Consider our discussion in Sect. 4.4 about the equilibrium contribution in public good games being inefficient. For simplicity, that section assumed that players' wealth was symmetric. In this exercise, we relax this assumption, allowing for asymmetric wealth, $w_i \neq w_j$.

(a) Set up the joint utility maximization problem to find the socially optimal aggregate and individual contributions to the public good.
(b) Under which conditions on wealth levels w_i and w_j every player submits a strictly positive contribution? Under which conditions only one player submits a positive contribution?
(c) Compare the socially optimal contributions that you found in part (a) against the equilibrium contributions in Sect. 4.4. Which contribution is larger? Interpret.

4.12. **Impure public good.**^{A} Consider the public good game in Sect. 4.4, but assume now that player i's return from aggregate donations is m_i while player j's is m_j, where $m_i, m_j \geq 0$ For simplicity, we consider symmetric wealth levels, so player i's utility function is

$$u_i(x_i, x_j) = (w - x_i)\sqrt{m_i X}.$$

The asymmetry in the return to the public good entails that its rivalry is impure because one player can benefit less from the same aggregate donations, X, than his rival does. In extreme settings, one player benefits from aggregate donations, if $m_i > 0$, while his rival does not benefit at all, if $m_j = 0$.

(a) Find player i's best response function, $x_i(x_j)$. How is it affected by a marginal increase in m_i?
(b) Find the NE contribution pair, (x_i^*, x_j^*).
(c) How is the equilibrium contribution, x_i^*, affected by a marginal increase in m_i? And by a marginal increase in m_j?

4.13. **Exploiting a common pool resource.**^{A} Consider a common pool resource such as a fishing ground, a forest, or an aquifer, exploited by $N = 2$ firms. In addition, consider that every firm takes prices as given (i.e., firms sell their appropriation in an international, perfectly competitive, market), where we can normalize this price to $p = \$1$, and every firm i's cost function is

$$C_i(q_i, q_j) = \frac{q_i(q_i + q_j)}{S},$$

where $q_i \geq 0$ denotes firm i's appropriation, $q_j \geq 0$ represents its rival's appropriation, and S is the initial stock of the resource.

(a) Find every firm i's marginal cost function. Interpret.
(c) Find every firm i's best response function, $q_i(q_j)$.
(c) How is $q_i(q_j)$ affected by an increase in S? Interpret.
(d) Find the equilibrium appropriation q_i^*.
(e) Find the appropriation levels that firms choose if they could coordinate their appropriation decisions to maximize their joint profits.

4.14. **A common pool resource with more general cost externalities.**^{B} Consider the setting in Exercise 4.13, but assume now that every firm i's cost function is

$$C_i(q_i, q_j) = \frac{q_i(q_i + \theta q_j)}{S},$$

where $\theta \geq 0$ denotes the severity of the cost externality. When $\theta = 0$, firm i's costs are unaffected by its rival's appropriation q_j; but when $\theta > 0$, firm i's costs are affected by q_j. Therefore, the setting in Exercise 4.12 can be interpreted as a special case of this (more general) model, where $\theta = 1$.

(a) Find every firm i's best response function, $q_i(q_j)$.
(b) How is $q_i(q_j)$ affected by parameter θ? Interpret.
(c) Find firms' appropriation in the NE of this game, q_i^*.
(d) How is q_i^* affected by parameter θ? Interpret.

4.15. **A common pool resource with asymmetric firms.**^{B} Consider the setting in Exercise 4.13 again, but assume that firm i faces cost functions $C_i(q_1, q_2) = \frac{\alpha_i q_i (q_i + q_j)}{S}$, where $\alpha_1 = 1$ and $\alpha_2 \in [0, 1]$. When $\alpha_2 = 1$, both firms are cost symmetric, as in Exercise 4.13; but otherwise firm 2 benefits from a cost advantage.

(a) Find each firm's best response function.
(b) Compare firms 1 and 2's best response functions.
(c) Find the NE appropriation pair $\left(q_1^*, q_2^*\right)$.
(d) Which firm appropriates more units in equilibrium?
(a) How is equilibrium appropriation affected by parameter α? Interpret.

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